

# Certiably Optimal Doppler Positioning with LEO Satellites

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# Why certifiably optimal?



Trapped at an unknown ocean.

Only access to communication LEO satellites

Can we guarantee where are we?

From Accuracy to Optimality

# LEO-PNT vs. LEO-COM



LEO-PNT  
With Navigation signal



LEO-COM  
With Signals of opportunity (SOOP)

LEO-COM with mega-constellation can provide **Doppler shift** for navigation.

# Why Doppler positioning?

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**ORIGINAL ARTICLE**

WILEY  ION

## Navigation using carrier Doppler shift from a LEO constellation: TRANSIT on steroids

Mark L. Psiaki 

[M. Psiaki, *NAVIGATION*, 2020]

LEO Doppler shift  
0.01 m/s



Positioning  
1-5 m

Comparable to GNSS!

# Doppler Positioning Model

$$\min_{\mathbf{p}_r, dt_r} \sum_{i=1}^N \left\| D_i - (\mathbf{p}_r - \mathbf{p}^s)^T \mathbf{v}^s / \rho_i - c \cdot dt_r \right\|_{\mathbf{Q}_\varepsilon}$$

3D position  
and clock shift rate

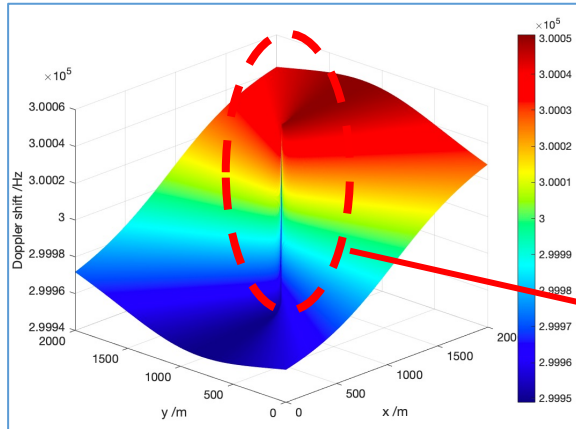
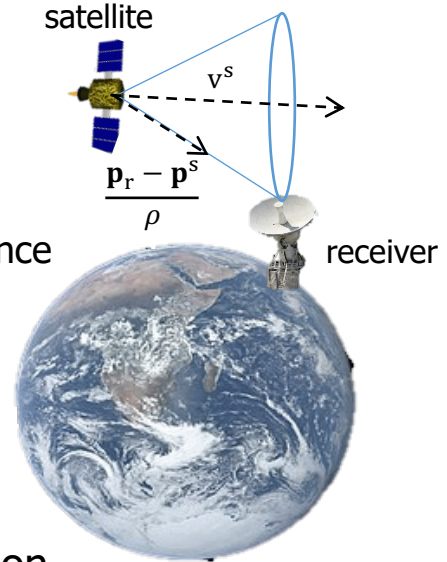
Doppler  
measurement

Satellite position  
and velocity

Geometric  
range

Light  
speed

Covariance  
matrix



Convexity requires **continuous** function.

The model is non-convex due to its uncontinuity near  $\mathbf{p}^s$ .

# Research Gaps

| LEO Doppler positioning   | Estimator                               | Initial Distance   |
|---|---|--------------------|
| Doppler PNT Review (M. Psiaki, <i>Navigation</i> , 2021)        | Batch Filter                            | $< 150 \text{ km}$ |
| Starlink (J. Khalife et al., <i>IEEE TAES</i> , 2022)           | Nonlinear Weighted Least Squares (NWLS) | $179 \text{ km}$   |
| Iridium-Next+Orbcomm (D. Wang et al., <i>IEEE TIM</i> , 2023)   | NWLS                                    | $< 100 \text{ km}$ |
| Simulation analysis (C. Shi et al., <i>GPS Solution</i> , 2023) | NWLS                                    | $< 300 \text{ km}$ |

Rely on exact initialization

Without Optimality guarantees

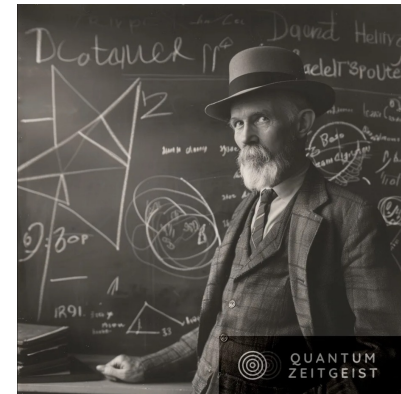
Is it possible to obtain a guaranteed **optimal** estimate?

# How to Guarantee Optimality?

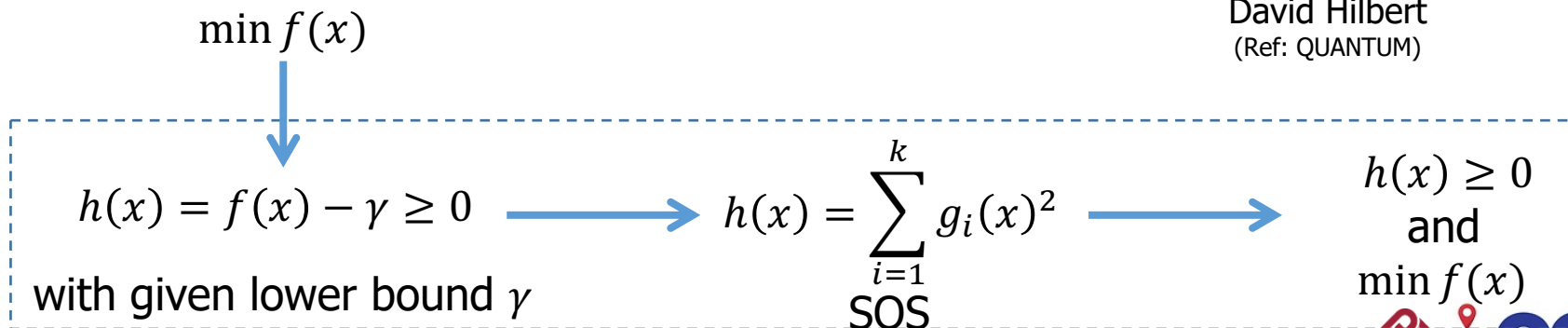
**Hilbert's 17<sup>th</sup> problem:** Can we rewrite a non-negative polynomial as a sum of squares (SOS)?

**Answer:** [J. Lasserre, 2001]

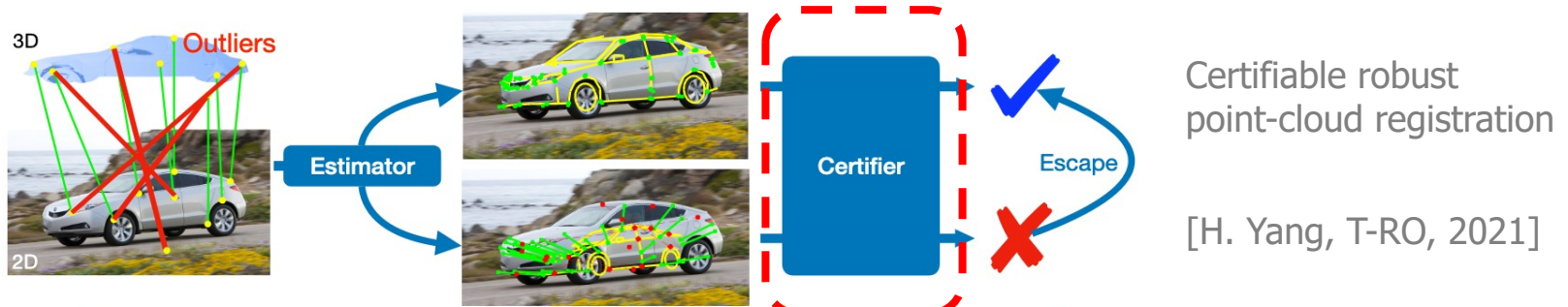
For some problems, we can try to **guarantee optimality** of  $f(x)$  by testing the **positivity constraint**  $h(x)$  using SOS.



David Hilbert  
(Ref: QUANTUM)

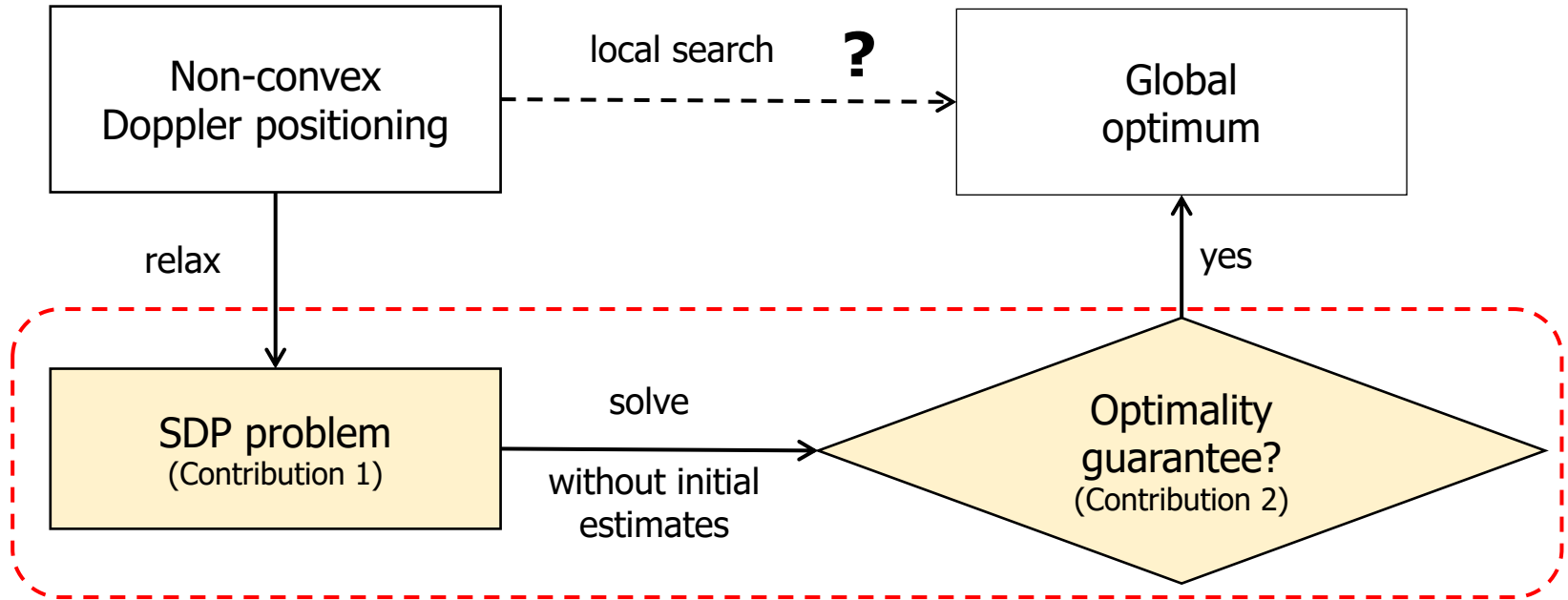


# Inspiration Example



We can use **semidefinite programming (SDP)** to search all SOS for Doppler positioning.

# Certiably optimal Doppler positioning



# Idea1: SDP Problem Formulation

Nonlinear weighted least squares (NWLS)

NWLS

$$\min_{\mathbf{p}_r, d\dot{t}_r} \sum_{i=1}^N \left\| D_i - (\mathbf{p}_r - \mathbf{p}^s)^T \mathbf{v}^s / \rho_i - c \cdot d\dot{t}_r \right\|_{\mathbf{Q}_\varepsilon}$$

Lifted state  $y = [p_r, d\dot{t}_r, \rho_1, \dots, \rho_N, \rho_1 d\dot{t}_r, \dots, \rho_N d\dot{t}_r]^T$ ;  $Y = \begin{bmatrix} yy^T & y \\ y^T & 1 \end{bmatrix}$

SOS

$$\begin{aligned} & \min_y \langle \mathbf{C}, \mathbf{Y} \rangle \\ & \text{s. t. } \langle \mathbf{A}_i, \mathbf{Y} \rangle = 0, \quad i = 1, \dots, 2N \end{aligned}$$

Ignore  
 $\text{rank}(\mathbf{Y}) = 1$

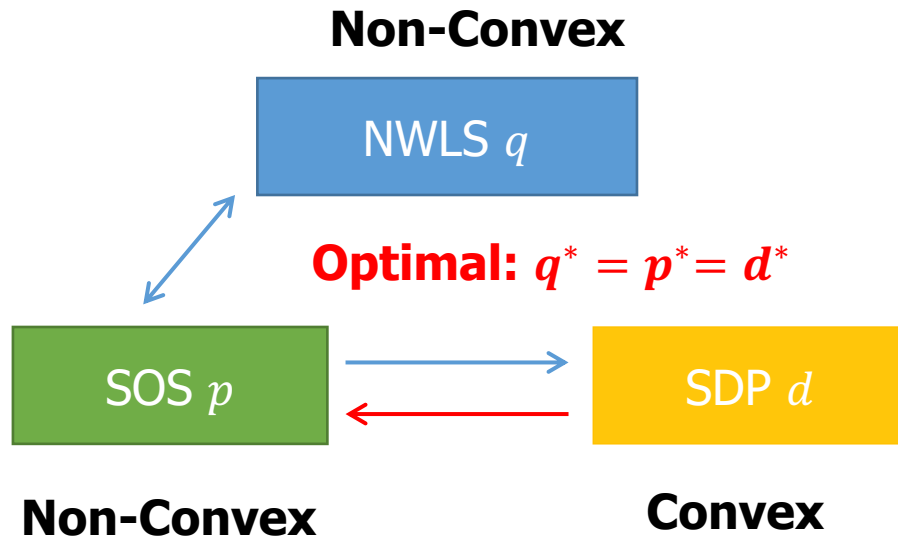
SDP

$$\begin{aligned} & \min_Y \langle \mathbf{C}, \mathbf{Y} \rangle \\ & \text{s. t. } \langle \mathbf{A}_i, \mathbf{Y} \rangle = 0, \quad i = 1, \dots, 2N \end{aligned}$$

Coefficient matrix  $\mathbf{C}$  and  $\mathbf{A}_i$

What is the **relationship** between them?

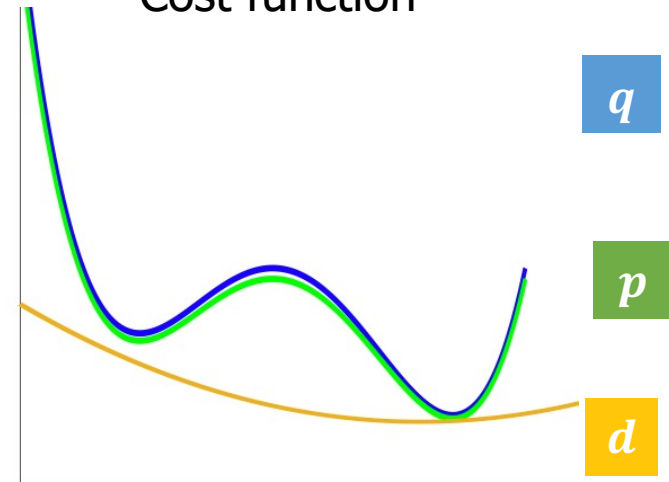
# Idea2: Optimality Guarantees



Karush-Kuhn-Tucker (KKT) condition

Guarantee the SDP exists.

Cost function

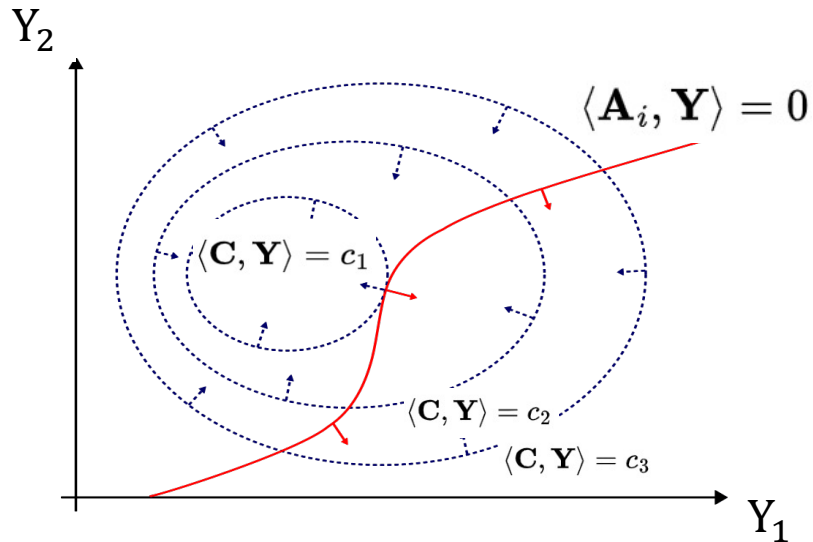


Rank-1 condition

Guarantee  $d^* = p^* = q^*$ .

# Idea2: Optimality Guarantees (Noiseless)

Karush-Kuhn-Tucker (KKT) conditions:



Sufficient condition for constrained problems.

Rank-1 condition:

$$Y = \begin{bmatrix} yy^T & y \\ y^T & 1 \end{bmatrix}$$



Resolve  
the unique  $y$  and  $p_r$

# Simulation Test

| Settings      | LEO Doppler positioning                           |
|---------------|---|
| Constellation | 49 satellites                                     |
| Orbit         | a square in 1500 * 1500 km,<br>at 800 km altitude |
| Frequency     | $1.626 \times 10^9$ Hz                            |
| Duration      | One epoch   |
| Ephemeris     | Simulated by $v = \sqrt{GM/r}$                    |

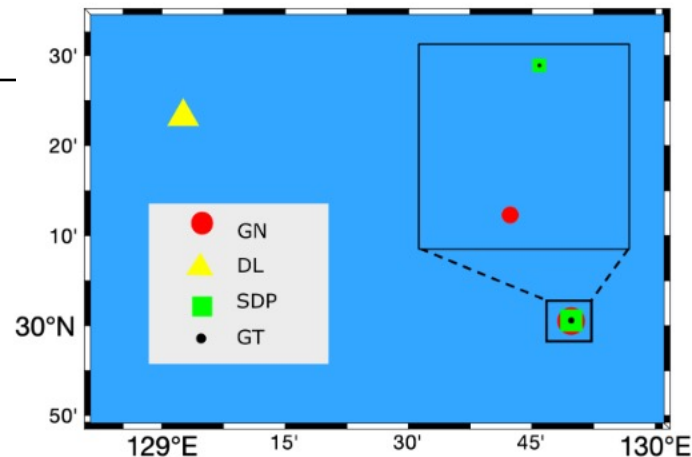
| Methods           | Initial guess | Local searching |
|-------------------|---------------|-----------------|
| Gauss-Newton (GN) | ✓             | ✓               |
| Dog-Leg (DL)      | ✓             | ✓               |

# Simulation Test

| Initial distance [km] | GN      | DL     | SDP  | SDP-GN | SDP-DL |
|-----------------------|---------|--------|------|--------|--------|
| 1                     | 1E-4    | 1E-4   | 0.71 | 1E-4   | 1E-4   |
| 10                    | 1E-4    | 1E-4   | 0.71 | 1E-4   | 1E-4   |
| 100                   | 1E-4    | 1E-4   | 0.71 | 1E-4   | 1E-4   |
| 580                   | 1839.60 | 91.65  | 0.71 | 1E-4   | 1E-4   |
| 1000                  | -       | 892.23 | 0.71 | 1E-4   | 1E-4   |

**GN, DL:** Fail at Local minima

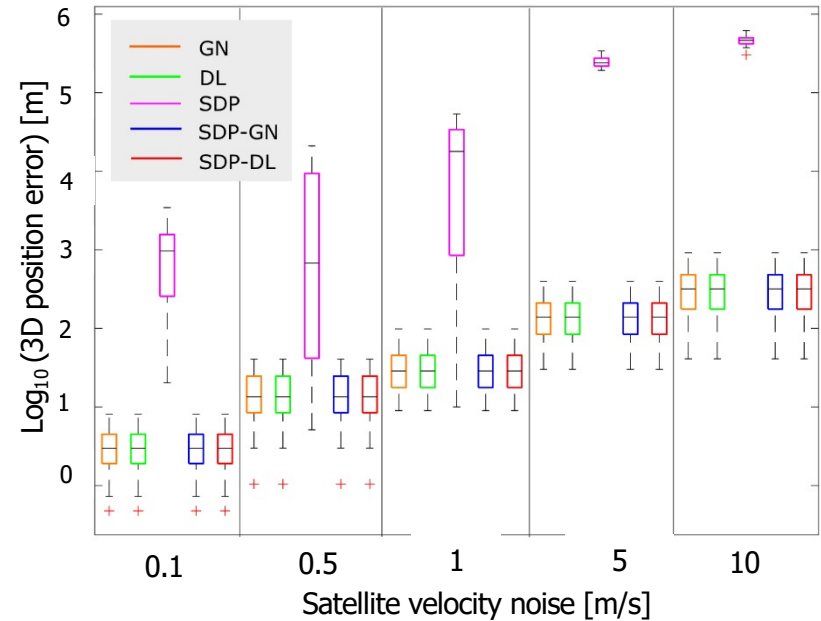
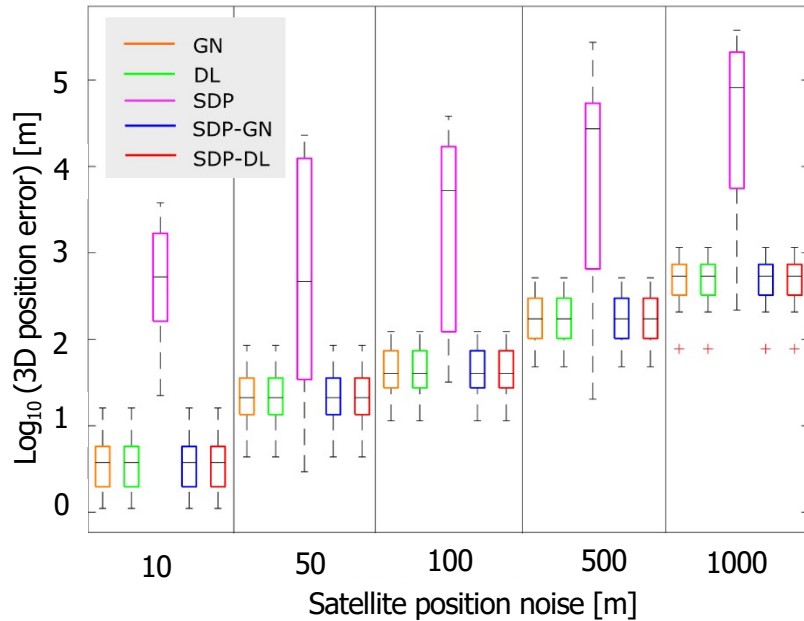
**SDP:** Global optimum



Initial distance=580 km

- 1) SDP is guaranteed **optimal** but lose accuracy due to CVX solver;
- 2) SDP+GN/DL is optimal and **accurate** using SDP as initialization.

# Simulation Test



**SDP + GN/DL** is not sensitive to **mild noise** from Two-Line Element ephemeris.

# Idea2: Optimality Guarantees (Noisy)

**Abadie Constraint Qualification (ACQ)**

1. Optimality still exists under mild noise disturbance; guarantees KKT conditions exist.

**Wyle's inequality**

guarantees the Rank-1 condition exists.

2. For noise bound, we need more **qualification condition**

to guarantee optimality.

Related to NWLS

Related to NWLS

$$\frac{1}{\sigma_N} \|\mathcal{G}\| \|\nabla f_{\theta}(\bar{\mathbf{y}})\| + \|\mathbf{F} - \mathbf{F}_{\theta}\| < \nu_4(\mathbf{F}_{\theta})$$

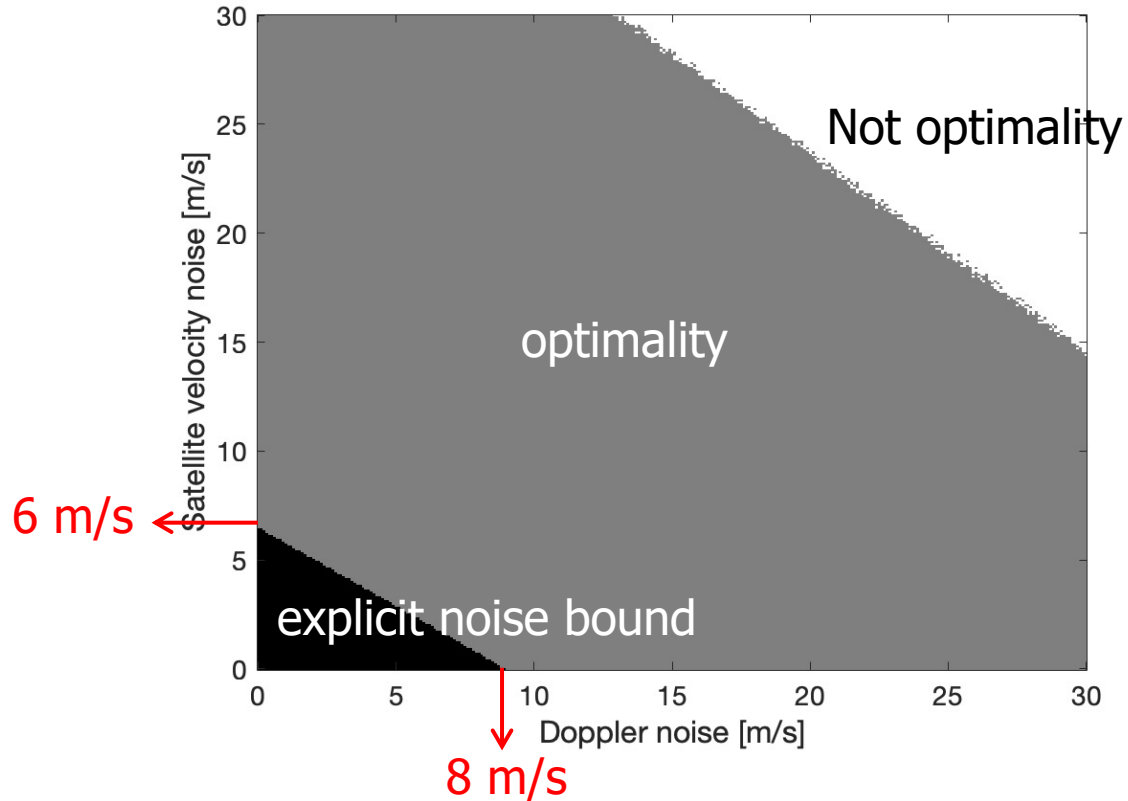
Related to NWLS

Related to SDP  
(noiseless and noisy)

Related to SDP

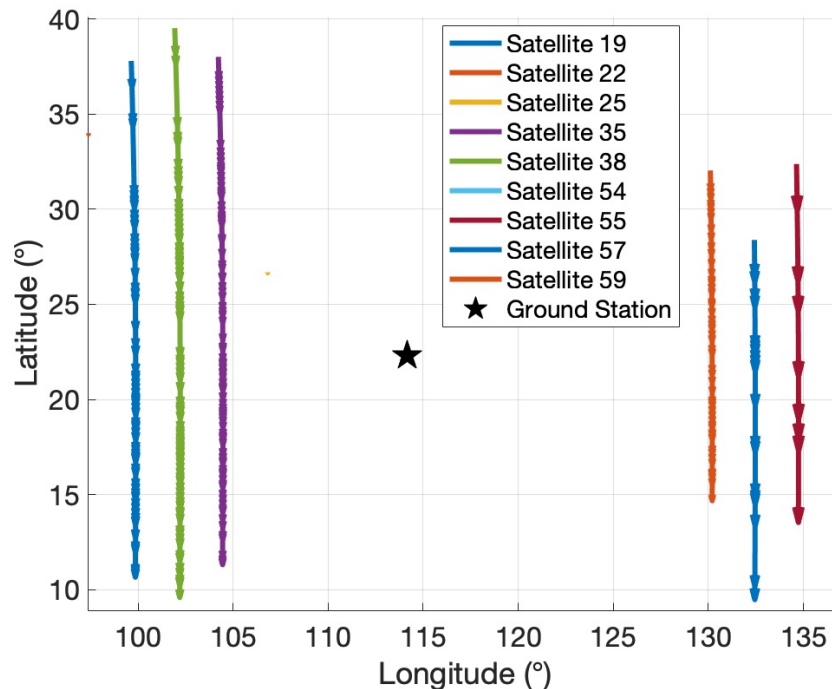
This inequality provides a **guaranteed noise bound** for real applications.

# Simulation Test



# Real Test

| Dataset                 | LEO Doppler positioning                      |
|-------------------------|--|
| Constellation           | Iridium-Next<br>(66 satellites in operation) |
| Orbit                   | Polar orbit ~780 km altitude                 |
| Frequency               | 1626270833 Hz                                |
| Visible Satellite       | 8  |
| Measurements & Duration | 436 measure in 35 s                          |
| Hardware                | NS210 + Geodetic antenna                     |
| Ephemeris               | TLE+SGP4                                     |



The data is provided by Dr. Bing Xu from HK-PolyU.

# Real Test

| Initial distance [km] | Gauss-Newton | Dog-Leg | SDP  | SDP-GN | SDP-DL |
|-----------------------|--------------|---------|------|--------|--------|
| 10                    | 0.13         | 0.13    | 0.14 | 0.13   | 0.13   |
| 100                   | 0.13         | 0.13    | 0.14 | 0.13   | 0.13   |
| 500                   | 0.13         | 0.13    | 0.14 | 0.13   | 0.13   |
| 1000                  | 2342.50      | 2069.00 | 0.14 | 0.13   | 0.13   |
| 5000                  | Fail         | 8445.10 | 0.14 | 0.13   | 0.13   |

**GN, DL: Fail** at Local minima

**SDP: Global optimum**

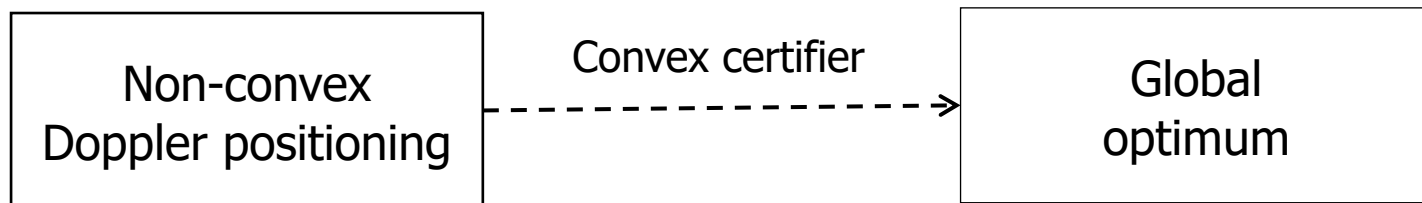


Initial distance=1000 km

- 1. Local searching methods may fail with **inexact initial position** (Local minima);
- 2. SDP method can provide **global optimality certificate** without **initial position** and can be used for initialization of GN/DL to increase the accuracy.

# Take-aways

- Certifiably optimal LEO Doppler positioning
  - Certifiably correct estimation using **global optimization** methods;
  - **Optimality analysis** for both noiseless and noisy cases.
  - Codes and data will be **open-sourced**.
- Meaning of this work



# Future work

- Limitation

1. Ignore **velocity** estimation for simplicity;
2. The SDP **optimization accuracy** is limited;

- Future work

1. Employ **moment sum-of-squares (SOS)** method to estimate velocity;
2. Develop a fast and accurate certifier **without SDP optimization.**

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# Open Source & Contact Information

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"Happy to give talks, or collaborate on  
LEO-PNT and multi-sensor navigation."



<https://github.com/Baoshan-Song/LEDSEER>  
(mathematical derivation, codes, data)

# Appendix A: local searching methods

- **State vector**

- $X = [p^s \quad v^s \quad dt^s]$

- **Doppler residual**

- $e_r^s = D_r^s - \left( \frac{\Delta p^T \Delta v}{\rho} - dt^s \right)$

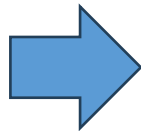
- $Z = [e_1^s \quad \dots \quad e_k^s]^T$

- **Jacobian matrix**

- $\frac{\partial e_r^s}{\partial p^s} = -\frac{\Delta v}{\rho} + \frac{\Delta p(\Delta p^T \Delta v)}{\rho^3} = \mathbf{m}$

- $\frac{\partial e_r^s}{\partial v^s} = -\frac{\Delta p}{\rho} = \mathbf{n}$

- $H = \begin{bmatrix} \mathbf{m}_1 & \mathbf{n}_1 & 1 \\ \vdots & \ddots & \vdots \\ \mathbf{m}_k & \mathbf{n}_k & 1 \end{bmatrix}$



$\Delta p = p^s - {}^E p_r$  : position vector

$\Delta v = v^s - {}^E v_r$  : velocity vector

$\rho = \|p^s - {}^E p_r\|$  : geometry range

$p^s, v^s$ : satellite position/velocity

$p_r, v_r$ : station position/velocity

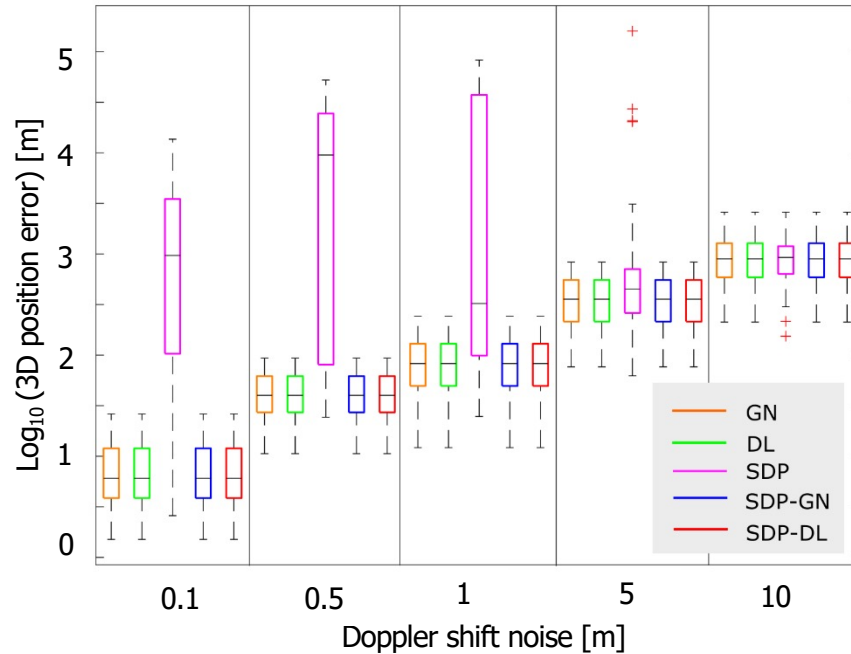
$dt^s$  : satellite clock drift rate

- **Weighted Least Square**

- $\Delta X = (H^T W H)^{-1} H^T Z$

- $\hat{X} = X + \Delta X$

# Appendix B: Simulation Test



SDP-GN/DL are not affected by mild Doppler noise.

